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Upstream influence of a dipole in rotating flow

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McIntyre (1972) has demonstrated that transient nonlinear self-interactions among the lee waves downstream of an obstacle in an axisymmetric, inviscid, rotating flow yield a columnar disturbance that moves upstream of the obstacle. This disturbance is calculated for a dipole, the moment of which increases slowly from 0 to a^3, in a tube of radius R as a function of $\delta = a/R$ and $K = 2\Omega R/U$ (Ω and U being angular and axial velocities in basic flow). The asymptotic limit $\delta \downarrow 0$ with $k \equiv K\delta$ fixed, which is relevant for typical laboratory configurations and for the abstraction of unbounded flow, yields: (*a*) an upstream velocity with an axial peak of $u_0 = 0{\cdot}011 k^5 \delta U$ and a first zero at a radius of $2{\cdot}8U/\Omega$; (*b*) an upstream energy flux of roughly $-\frac{3}{2}\pi\rho U^2 R^2 u_0$; (*c*) an upstream impulse flux of $0{\cdot}20D$ (D = wave drag on dipole). The results (*a*) and (*b*), albeit based on the hypothesis of small disturbances, suggest that nonlinear self-interactions among the lee waves could be responsible for upstream blocking. The results (*a*) and (*c*) imply that, although upstream influence is absent from an unbounded ($\delta = 0$) flow in the sense that the axial velocity vanishes like δ, it is present in the sense that approximately one-fifth of the impulse flux associated with the wave drag appears in the upstream flow.

1. Introduction

Benjamin (1970) has argued, and McIntyre (1972) has demonstrated, that transient nonlinear interactions in an inviscid, unseparated, axisymmetric, rotating flow downstream of an obstacle in a cylindrical tube yield columnar disturbances (I use *columnar disturbance* in the same sense as McIntyre) that move upstream of the obstacle and negate Long's (1953) hypothesis of no upstream influence (which implies a linear equation for the stream function) for disturbances of finite amplitude. On the other hand, McIntyre has shown that these nonlinear interactions are evanescent in an (externally) unbounded rotating flow, in which case Long's hypothesis remains valid. Introducing

$$k = 2\Omega a/U \equiv K\delta \tag{1.1}$$

and

$$\delta = a/R, \tag{1.2}$$

where Ω and U are the angular and axial velocities in the basic flow, a is a characteristic lateral radius of the obstacle, and R is the radius of the tube, we may state that Long's hypothesis remains valid† in either of the limits

(i) $\delta \downarrow 0$ with K fixed,

or (ii) $\delta \downarrow 0$ with k fixed,

† This validity may not be uniform as $t \to \infty$; see McIntyre (1972, § 7).

corresponding to either (i) infinitesimal disturbances in bounded flow or (ii) finite disturbances in unbounded flow. Of these two limits, the latter appears to be the more relevant to typical laboratory experiments (e.g., Maxworthy 1970), in which δ is small and K is large.

The hypothesis of unseparated flow† permits the obstacle to be described by an axial distribution of dipoles provided that the observed upstream-separation bubble is incorporated into the theoretical model by regarding it as part of the obstacle (Miles 1969, 1970*a*; Maxworthy 1970). The determination of this distribution for a prescribed obstacle generally requires the solution of an integral equation (Miles 1969) and involves additional nonlinear effects through the boundary conditions (the latter effects are essentially local and are distinct from nonlinear effects in the far field that give rise to the columnar disturbances; McIntyre gives a fuller discussion of this point); however, the far field of the obstacle may be represented by a dipole singularity for moderate $\not{k}$ (say $\not{k} \leqslant 2$). The strength of this singularity, designated as the *dipole moment* $\equiv R^3\epsilon$, is proportional to a^2l for a slender body of lateral radius a and length l and to a^3 for a fat body of lateral radius a; in particular, $\epsilon \uparrow (a/R)^3$ for a sphere of radius a as $\not{k} \downarrow 0$.

Against this background, we consider an axisymmetric inviscid flow that is initially ($t \leqslant 0$) uniform, with axial velocity U and angular velocity Ω, and give an asymptotic ($t \rightarrow \infty$) description of the columnar disturbances induced by a dipole, the moment of which increases uniformly from 0 to $R^3\epsilon$ on the time scale $R/(U\sigma)$, where both ϵ and σ are small. This description (in §§ 2 and 3) is essentially similar to that given by McIntyre (1972) for an impulsively started body, and we therefore omit its detailed derivation (although this derivation differs significantly from that given by McIntyre). We then go on (in §§ 4 and 5) to develop asymptotic approximations in the limit (ii). These results go considerably beyond those given by McIntyre [although they rest directly on his basic work, as well as the earlier work of Benjamin (1970)], and we therefore develop them in more detail.

Formulation

Choosing R and U as scales of length and velocity, we pose the velocity and vorticity vectors in the forms

$$\mathbf{v} = U(1, 0, \tfrac{1}{2}Kr) + Ur^{-1}(\psi_r, -\psi_x, K\gamma) \tag{1.3a}$$

and

$$\nabla \times \mathbf{v} = 2\Omega(1, 0, 0) + 2\Omega r^{-1}(\gamma_r, -\gamma_x, K\varpi), \tag{1.3b}$$

† The complete theoretical description of separated flow poses an intractable problem; however, an investigation of viscous effects in an unbounded fluid suggests that such a flow may be described asymptotically by an oseenlet singularity. The inner limit, $E \boxminus 2\Omega\nu/U^2 \downarrow 0$ with co-ordinates fixed, of the resulting solution yields an upstream velocity with the same radial distribution as the dipole singularity and an amplitude proportional to the viscous drag (Miles 1970*b*). Comparison with Maxworthy's (1970) experiments suggests that this amplitude is small for moderate values of $\not{k}$, $\delta \ll 1$, and $E \ll 1$, but the possibility remains that the columnar disturbance associated with viscous separation could be comparable with, or even dominate, that associated with nonlinear interactions.

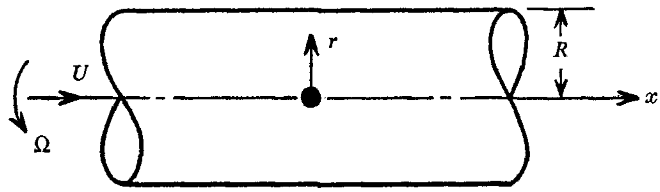

FIGURE 1. Sketch of dipole in rotating flow.

where the triad (–, –, –) comprises the axial, radial, and azimuthal components of a vector, x and r are dimensionless cylindrical co-ordinates referred to the length scale R, $UR^2\psi$ is the perturbation stream function, $KUR\gamma$ is the azimuthal circulation relative to the basic flow, $K^2U\varpi$ is the radial moment of the azimuthal vorticity (subsequently to be related to the axial impulse), and K is defined by (1.1). We also introduce the column matrices

$$\boldsymbol{\psi}(x,r,t) = \{\psi, \gamma, \varpi\}, \quad \mathbf{I} = \{1,1,1\} \tag{1.4a,b}$$

and note that Long's hypothesis would imply a solution of the form $\boldsymbol{\psi} = \mathbf{I}\psi$. The resulting equations of motion then may be posed in the matrix form (cf. Goldstein 1938, p. 115)

$$\begin{bmatrix} \partial_x^2 + r\,\partial_r r^{-1}\partial_r & 0 & K^2 \\ -\partial_x & \partial_t + \partial_x & 0 \\ 0 & -\partial_x & \partial_t + \partial_x \end{bmatrix} \boldsymbol{\psi} = \mathscr{N}, \tag{1.5}$$

where
$$\mathscr{N} = \left\{0, \frac{1}{r}\frac{\partial(\psi,\gamma)}{\partial(x,r)}, \quad \frac{1}{r}\frac{\partial(\psi,\varpi)}{\partial(x,r)} + \frac{2}{r^2}(\gamma\gamma_x - \varpi\psi_x)\right\} \tag{1.6}$$

is a column matrix that comprises the nonlinear terms. The initial condition is

$$\boldsymbol{\psi}(x,r,0) = 0. \tag{1.7}$$

The boundary conditions are

$$\psi(x,r,t) \to -\epsilon\mathscr{d}(x)f(t), \quad \psi_r \to 0 \quad (r\downarrow 0), \tag{1.8a,b}$$

$$\partial_x\psi(x,1,t) = 0, \quad \psi(\pm\infty, r, t) = 0, \tag{1.8c,d}$$

where $\mathscr{d}(x)$ is Dirac's delta function, $R^3\epsilon$ is the dipole moment (see above), and $f(t)$ is a slowly varying function that satisfies

$$f(t) = 0 \quad (t \leqslant 0), \quad f(t) \sim 1 + O(1/t) \quad (t\to\infty), \tag{1.9a,b}$$

and
$$\dot{f}(t) = O(\sigma) \tag{1.9c}$$

uniformly with respect to t.

2. First approximation

The solution of (1.5)–(1.8) may be developed in powers of ϵ, starting from the first approximation, in which $\mathscr{N} = O(\epsilon^2)$ is neglected on the right-hand side of (1.5), and proceeding by iteration. Given the nth approximation to $\mathscr{N}$, the $(n+1)$th approximation to $\boldsymbol{\psi}$ may be obtained with the aid of Fourier, finite-Hankel, and Laplace transforms with respect to x, r, and t. The details of the solution depend on the order in which the Fourier and Laplace transforms are inverted (the inversion of the finite-Hankel transform leads directly to a Fourier–Bessel series independently of the order of the Fourier and Laplace inversions), and some type of asymptotic approximation is essential for the display of explicit results (which, in the present context, are required only for large $|x|$).

McIntyre constructs the first (linearized) approximation by first inverting the Laplace transform exactly and then considering the asymptotic properties of the Fourier integral as $|x| \to \infty$. An alternative procedure, which is more economical but also more heuristic, is to invoke the hypotheses (1.9b, c), which imply that the Laplace-transform parameter s is $O(\sigma)$, determine the poles of the Fourier transform through $O(\sigma)$, and then invert the Fourier and Laplace transforms in that order. This latter procedure yields the first approximation in the form

$$\psi^{(1)}(x,r,t) = r\left[\sum_1^N \Psi_n^{(w)}(x, t-\mu_n x) + \sum_{N+1}^\infty \Psi_n^{(e)}(x, t+|\mu_n|\, x) + \sum_{N+1}^\infty \Psi_n^{(-)}(t-\mu_n^{(-)}x) + \sum_1^\infty \Psi_n^{(+)}(t-\mu_n^{(+)}x)\right] J_1(\beta_n r) \quad (x > 0) \tag{2.1a}$$

$$= r\left[\sum_{N+1}^\infty \Psi_n^{(e)}(x, t+|\mu_n|\, x) - \sum_1^N \Psi_n^{(-)}(t+\mu_n^{(-)}x)\right] J_1(\beta_n r) \quad (x < 0), \tag{2.1b}$$

where the β_n are determined by

$$J_1(\beta_n) = 0 \quad (0 < \beta_1 < \beta_2 < \ldots), \tag{2.2}$$

N, the number of lee-wave modes, is determined by

$$\beta_N < K < \beta_{N+1}, \tag{2.3}$$

the reciprocal wave speeds are given by

$$\mu_n = K^2/(K^2-\beta_n^2) \equiv 1/c_n, \quad \mu_n^{(\pm)} = \beta_n |K \pm \beta_n|^{-1} \equiv 1/c_n^{(\pm)}, \tag{2.4a, b}$$

$\Psi_n^{(w)}$ represents a lee wave with the axial wavenumber

$$\kappa_n = (K^2-\beta_n^2)^{\frac{1}{2}} \tag{2.5}$$

that advances downstream with the wave-front speed c_n, $\Psi_n^{(e)}$ represents an exponentially decaying disturbance that extends indefinitely downstream (in the present approximation) and advances with the wave-front speed $1/|c_n|$ in the upstream direction, $\Psi_n^{(+)}$ represents a transient that is evanescent as $t \to \infty$ and advances downstream with the wave-front speed $c_n^{(+)}$, and $\Psi_n^{(-)}$ represents a similar transient that advances either upstream ($\beta < K$) or downstream ($\beta > K$) with the wave-front speed $c_n^{(-)}$.

The explicit determination of the Ψ_n is straightforward; however, McIntyre's analysis implies that, under (1.9), only the lee-wave component of $\psi^{(1)}$,

$$\psi^{(w)}(x,r,t) = H(x)\, r \sum_1^N \Psi_n^{(w)}(x, t-\mu_n x)\, J_1(\beta_n r), \tag{2.6}$$

contributes to the next term in the development, say $\psi^{(2)}$, at $O(\epsilon^2)$. Moreover, if $\psi^{(w)}$ is resolved into $\mathrm{I}\psi^{(w)}$ and $\psi^{(w)} - \mathrm{I}\psi^{(w)}$, terms that are $O(\sigma)$ are significant only in the latter component (since a solution of the form $\mathbf{I}\psi$ renders $\mathcal{N} \equiv 0$). Carrying out the solution at this approximation, we obtain

$$\psi_n^{(w)} = \mathscr{R}\big(A_n[\mathbf{I}f(t-\mu_n x) + i\kappa_n^{-1}\{0, 1, 2\}\dot{f}(t-\mu_n x)]\exp[i(\kappa_n x - \tfrac{1}{2}\pi)]\big), \tag{2.7}$$

where
$$A_n = 2\epsilon\beta_n(\kappa_n J_{0n}^2)^{-1} \tag{2.8}$$

is the amplitude of the nth lee-wave mode,

$$J_{0n} = J_0(\beta_n), \tag{2.9}$$

and the error factor is $1+O(\sigma)$, both for the term proportional to $\mathbf{I}$ and, independently, for the term proportional to $\{0, 1, 2\}$. We note that A_n is complex for a dipole distribution; see (4.14) *et seq.*

We emphasize that neither (2.1) nor (2.6) provides a detailed description of the wave fronts, whereas McIntyre (1972) does provide such a description.

3. Second approximation

Substituting (2.6) into (1.6) and retaining only the slowly changing (in both x and t) columnar disturbances that arise from the quadratic interactions between the lee-wave components of like index, we obtain

$$\mathcal{N} = H(x)\, r \sum_1^N |A_n|^2 \beta_n\{0, \tfrac{1}{2}J_0(\beta_n r), J_1'(\beta_n r)\} J_1(\beta_n r)\, \partial_t f^2(t-\mu_n x). \tag{3.1}$$

The remaining contributions to $\mathcal{N}$ of the first approximation, equation (2.1), contribute to $\psi^{(2)}$ only at $O(\sigma\epsilon^2)$.

Substituting (3.1) into the right-hand side of (1.5) and then repeating the integral-transform solution of (1.5)–(1.9), we obtain the second approximation (which is equivalent to that given by McIntyre in his appendix A):

$$\psi = \psi^{(1)} + \psi^{(2)}, \tag{3.2}$$

where
$$\psi^{(2)} = r\sum_{m=1}^{N}\Bigg[\sum_{n=1}^{\infty}(\mathbf{A}_{mn}^{(+)} + \mathbf{A}_{mn}^{(-)}) f^2(t-\mu_m x) - \sum_{n=1}^{\infty}\mathbf{A}_{mn}^{(+)} f^2(t-\mu_n^{(+)}x) - \sum_{n=N+1}^{\infty}\mathbf{A}_{mn}^{(-)} f^2(t-\mu_n^{(-)}x)\Bigg] J_1(\beta_n r) \quad (x>0) \tag{3.3a}$$

$$= r\sum_{m=1}^{N}\sum_{n=1}^{N}\mathbf{A}_{mn}^{(-)} f^2(t+\mu_n^{(-)}x)\, J_1(\beta_n r) \quad (x<0), \tag{3.3b}$$

$$\mathbf{A}_{mn}^{(\pm)} = \frac{|A_m|^2 (K^2-\beta_m^2)(\beta_{mn}\beta_n \mp 2\alpha_{mn}K)}{8(K^3 \pm \beta_m^2\beta_n)}\left\{\frac{K}{\beta_n}, \mp 1, \frac{\beta_n}{K}\right\}, \tag{3.4a}$$

$$\alpha_{mn} = -2\beta_n J_{0n}^{-2}\int_0^1 J_1^2(\beta_m r)\, J_0(\beta_n r)\, r\, dr, \tag{3.4b}$$

$$\beta_{mn} = 2\beta_n J_{0n}^{-2}\int_0^1 J_0^2(\beta_m r)\, J_0(\beta_n r)\, r\, dr, \tag{3.4c}$$

and the reciprocal wave speeds are given by (2.4). McIntyre (1972) tabulates α_{mn} and β_{mn} for $m, n = 1(1)\,6$. We note that

$$\beta_{nn} = -2\alpha_{nn} = \tfrac{8}{3}J_{0n}^{-2}\int_0^1 J_1^3(\beta_n r)\, dr \tag{3.5}$$

and that asymptotic approximations for $\beta_m, \beta_n \gg 1$ may be obtained by replacing the upper limits of the integrals in (3.4) by ∞ and invoking the known results for the infinite integrals (Watson 1945, § 13.46). Invoking also

$$\beta_n \sim n'\pi, \quad J_{0n} \sim (-)^n \pi^{-1} (2/n')^{\frac{1}{2}}, \quad n' \equiv n + \tfrac{1}{4}, \tag{3.6}$$

we obtain

$$\beta_{mn} \sim -[1 - \tfrac{1}{2}(n'/m')^2]^{-1} \alpha_{mn} \sim [(m'/n')^2 - \tfrac{1}{4}]^{-\frac{1}{2}} \quad (m' > \tfrac{1}{2}n'), \tag{3.7}$$

which are within 1% of the exact results for $m, n > 6$; both β_{mn} and α_{mn} are $O(1/m')$ if $m' < \frac{1}{2}n'$.

We remark that (3.3a) is indeterminate if $\mu_m = \mu_n^{(-)}$, which can occur only if $\beta_n > K$ and implies coalescence of the corresponding fronts. Resolving the indeterminacy, we obtain

$$\mathrm{A}_{mn}^{(-)}[f^2(t-\mu_m x) - f^2(t-\mu_n^{(-)}x)]$$
$$\rightarrow \frac{|A_m|^2 (\beta_{mn}\beta_n + 2\alpha_{mn} K)}{8(\beta_n - K)} \left\{\frac{K}{\beta_n}, 1, \frac{\beta_n}{K}\right\} x \partial_t f^2(t - \mu_n^{(-)} x) \quad (\beta_m^2 \rightarrow K^3/\beta_n < K^2). \tag{3.8}$$

Letting $t \rightarrow \infty$ in (3.3) and invoking (1.9b), we find that the disturbances in $0 < x < c_N^{(-)} t$ mutually cancel if $\beta_n > K$, whilst those in $0 < x < c_N t$ yield a disturbance identical with that in $x < 0$ if $\beta_n < K$, such that the columnar component of $\boldsymbol{\psi}$ is given asymptotically by

$$\boldsymbol{\psi}^{(2)} \sim r \sum_{m=1}^{N} \sum_{n=1}^{N} \mathrm{A}_{mn}^{(-)} J_1(\beta_n r) \quad (t \rightarrow \infty, |x| \text{ fixed}). \tag{3.9}$$

We note, however, that (3.9) is not uniformly valid as $x \rightarrow \infty$ if $\beta_m^2 \beta_n - K^3$ is small in consequence of the x factor in (3.8).

4. Upstream influence

The total disturbance in the ordered limit $t \rightarrow \infty$, $x \rightarrow -\infty$ is given by (3.9). Substituting the ψ component of (3.9) into (1.3a), we obtain

$$\mathbf{v} \sim \mathbf{v}^{(0)} - \epsilon^2 U \sum_{1}^{N} \mathscr{U}_n \big(J_0(\beta_n r), 0, J_1(\beta_n r)\big) \quad (t \rightarrow \infty, x \rightarrow -\infty), \tag{4.1}$$

where $\mathbf{v}^{(0)}$ is the velocity in the basic flow, as given by the first term on the right-hand side of (1.3a), and

$$\mathscr{U}_n = -\tfrac{1}{2} K \sum_{m=1}^{N} \beta_m^2 (K^3 - \beta_m^2 \beta_n)^{-1} J_{0m}^{-4} (2\alpha_{mn} K + \beta_{mn} \beta_n). \tag{4.2}$$

There is no net mass flux associated with $\boldsymbol{\psi}^{(2)}$, as is already implicit in the hypothesis of a dipole representation of a body; however, there is a net flux of both energy and impulse. The perturbation energy per unit length is given by

$$\partial E/\partial x = \pi \rho R^3 \int_0^1 (|\mathbf{v}|^2 - |\mathbf{v}_0|^2)\, r\, dr. \tag{4.3}$$

Substituting (1.3*a*) into (4.3) and invoking (1.8*c*) to eliminate the integral of the first-order term in ψ_r, we obtain

$$\partial E/\partial x = \pi\rho U^2 R^3 \int_0^1 [K^2\gamma + r^{-2}(\psi_x^2 + \psi_r^2 + K^2\gamma^2)]\, r\, dr. \qquad (4.4)$$

Conservation of angular momentum implies that the volume integral of $K^2\gamma$ throughout the entire tube ($0 \leqslant r \leqslant 1$, $-\infty < x < \infty$) must vanish identically. A detailed calculation confirms this fact within the present approximation and reveals that the integrated angular momentum, and hence energy, in $x > c_N t$ increases at an asymptotically constant rate. It follows that the steady-state solution is inadequate for a complete discussion of the energy balance in the flow. This *caveat* is important in interpreting the subsequent results.†

Substituting the γ component of (3.3*b*) into (4.4), carrying out the integral over the upstream domain according to

$$\int_{-t/\mu^{(-)}}^{0} f(t + \mu^{(-)} x)\, dx = c^{(-)} \int_0^t f(\tau)\, d\tau, \qquad (4.5)$$

differentiating with respect to real time,

$$T = Rt/U, \qquad (4.6)$$

letting $t \to \infty$, and invoking (1.9*b*), we obtain the rate of change of the upstream perturbation energy, say $E^{(-)}$, within $1 + O(\epsilon^2)$ in the form

$$dE^{(-)}/dT \sim 2\epsilon^2 \dot{E}_0 K \sum_1^N \mathscr{U}_n (K - \beta_n)\, \beta_n^{-2} J_{0n} \equiv \epsilon^2 \dot{E}_0 \mathscr{E}^{(-)} \quad (t \to \infty), \qquad (4.7)$$

where

$$\dot{E}_0 = \tfrac{1}{2}\pi\rho U^3 R^2 \qquad (4.8)$$

is a measure of the energy flux in the basic flow.

The total perturbation energy is dominated by the lee-wave energy, say $E^{(w)}$, in consequence of the aforementioned fact that the volume integral of γ vanishes identically. Substituting ψ and γ from (2.7) into (4.4) and proceeding as above, we obtain

$$dE^{(w)}/dT = UD \sim 4\epsilon^2 \dot{E}_0 \sum_1^N (\beta_n / J_{0n})^2 \equiv \epsilon^2 \dot{E}_0 \mathscr{E}^{(w)} \quad (t \to \infty), \qquad (4.9)$$

where D is the wave drag.

Benjamin (1970, § 3) has shown that

$$D = dP/dT, \qquad (4.10)$$

where P is the axial component of the impulse. In his notation, only $(P_V)_x$ contributes to dP/dT (there is a contribution to P from the surface integral over the body, Benjamin's equation (3.14), but this is asymptotically constant); in the present notation, Benjamin's equation (3.15) yields the impulse density

$$dP/dV = -\tfrac{1}{2}\rho U K^2 \varpi, \qquad (4.11)$$

† Dr McIntyre has remarked (in a private communication) that the difficulties associated with the energy balance are at least partially a consequence of the choice of a fixed, rather than a rotating, reference frame.

where ϖ is defined by (1.3*b*) and (1.4*a*). Benjamin argues from (4.10) that upstream influence is a necessary concomitant of wave drag; but, as McIntyre (1972) points out, the upstream contribution to P may vanish identically, as in two-dimensional stratified flow if only the dominant mode is propagated. It therefore is of some interest to calculate the upstream component of P, say $P^{(-)}$, explicitly. Proceeding as in the calculation of $E^{(-)}$, we obtain

$$dP^{(-)}/dT \sim -2\epsilon^2(\dot{E}_0/U)\sum_1^N \mathscr{U}_n(K-\beta_n)\beta_n^{-1}J_{0n} \equiv \epsilon^2(\dot{E}_0/U)\mathscr{P}^{(-)} \quad (t\to\infty). \tag{4.12}$$

The total-impulse parameter corresponding to $\mathscr{P}^{(-)}$ is $\mathscr{P} \equiv \mathscr{E}^{(w)}$.

The preceding results simplify as follows if $N = 1$ (so that only the dominant mode is propagated):

$$\mathscr{U}_1 = 98{\cdot}3K_1(1+K_1+K_1^2)^{-1} \quad (K_1 = K/\beta_1 = 0{\cdot}261K), \tag{4.13a}$$

$$\mathscr{E}^{(w)} = 362, \quad \mathscr{E}^{(-)} = -0{\cdot}806K_1(K_1-1)\mathscr{U}_1, \quad \mathscr{P}^{(-)} = 0{\cdot}806(K_1-1)\mathscr{U}_1. \tag{4.13b, c, d}$$

Axial distribution of dipoles

The asymptotic results ($t \to \infty$) may be generalized to a dipole distribution by redefining $d(x)$ in (1.8*a*) as the dipole density, normalized to unit moment. It then is necessary only to insert the distribution factor

$$\mathscr{D}_n = \int_{-\infty}^{\infty} d(x)\, e^{-i\kappa_n x}\, dx \tag{4.14}$$

on the right-hand side of (2.8), and to insert $|\mathscr{D}_m|^2$, $|\mathscr{D}_n|^2$, and $|\mathscr{D}_1|^2$ in the right-hand sides of (4.2), (4.9), and (4.13*a, b*), respectively. The approximation of a finite obstacle by a dipole implies $\mathscr{D}_n = 1$ for all n. A necessary condition for this approximation is

$$k \equiv 2\Omega l/U \ll 1, \tag{4.15}$$

where l is the length of the obstacle.

5. Numerical results

The following results are based on tabulated values of β_n and J_{0n} and numerical integration of α_{mn} and β_{mn} for $m, n \leqslant 6$ and on (3.6) and (3.7) for $m, n > 7$. The various parameters exhibit a staircase-like growth with K, with finite jumps at $K = \beta_m$ and slow monotonic variations in absolute value as K increases from β_m+ to $\beta_{m+1}-$ (see figures 3 and 5). The results for $N = 1$, which are given by (4.13) and are on a significantly smaller scale than those for $N > 2$, are plotted in figure 2. The minima and maxima of the $\mathscr{U}_n$ for $N = 1(1)12$ are tabulated in table 1 (intermediate values may be determined by linear interpolation on a log–log graph). Calculations up to $N = 20$ suggest that $\mathscr{U}_n > 0$ for $n \leqslant M(\beta)$ and $\mathscr{U}_n < 0$ for $n > M$, where M increases with N (table 1 yields $M = N-1$ for $2 \leqslant N \leqslant 9$).

K/π \ n	1	2	3	4	5	6	7	8	9	10	11	12
1·220 −	0											
1·220 +	14·80											
2·233 −	1·17											
2·233 +	29·60	−11·70										
3·238 −	3·03	−0·53										
3·238 +	15·23	21·98	−8·22									
4·241 −	3·27	3·13	−0·19									
4·241 +	10·03	15·14	21·19	−17·55								
5·243 −	3·12	3·98	3·78	−3·21								
5·243 +	7·41	11·43	16·61	14·62	−13·19							
6·244 −	2·89	4·06	5·01	2·50	−2·37							
6·244 +	5·85	9·13	13·56	14·17	14·03	−19·66						
7·25 −	2·64	3·88	5·27	4·56	2·81	−5·55						
7·25 +	4·62	7·70	11·20	12·91	13·94	8·81	−14·65					
8·25 −	2·34	3·74	5·14	5·37	4·90	1·04	−4·02					
8·25 +	3·85	6·63	9·59	11·58	13·09	11·49	9·04	−22·43				
9·25 −	2·11	3·53	4·91	5·58	5·77	4·01	1·55	−8·30				
9·25 +	3·31	5·80	8·38	10·37	12·05	11·94	11·36	3·61	−16·15			
10·25 −	1·94	3·32	4·66	5·53	6·07	5·40	4·25	−1·02	−5·63			
10·25 +	2·90	5·15	7·43	9·35	11·03	11·63	11·89	8·19	5·33	−22·91		
11·25 −	1·79	3·12	4·40	5·37	6·10	6·03	5·61	2·69	0·21	−9·84		
11·25 +	2·59	4·62	6·67	8·48	10·12	11·05	11·73	10·01	8·86	0·29	−17·12	
12·25 −	1·67	2·93	4·16	5·16	5·98	6·26	6·27	4·67	3·32	−2·56	−6·97	
12·25 +	2·34	4·19	6·05	7·74	9·31	10·39	11·29	10·64	10·33	5·60	2·45	−23·20

TABLE 1. The normalized Fourier–Bessel coefficients $(\pi/K)^4\, \mathscr{U}_n$ for the upstream velocity given by (4.1).

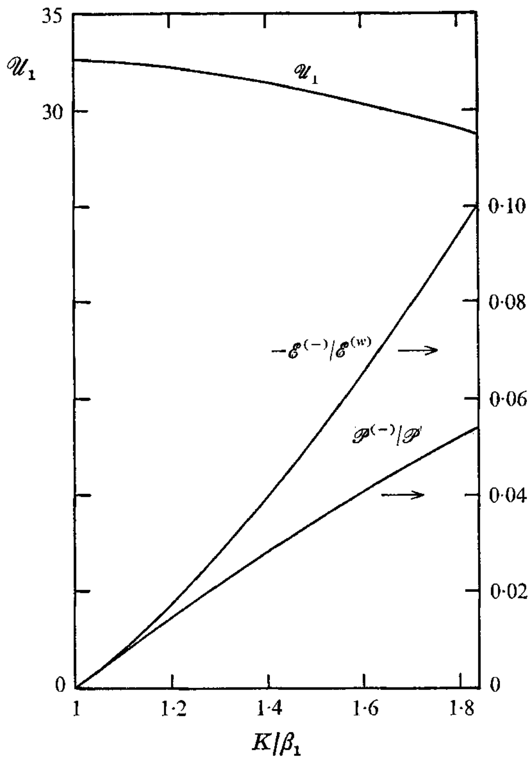

FIGURE 2. The parameters $\mathscr{U}_1$, $\mathscr{E}^{(-)}$, and $\mathscr{P}^{(-)}$ for $N = 1$, as given by (4.13); $\mathscr{P} = \mathscr{E}^{(w)} = 362$.

The relative axial velocity in the direction of motion of the dipole, as determined from (4.1), is

$$u(r) = \epsilon^2 U \sum_1^N \mathscr{U}_n J_0(\beta_n r), \tag{5.1a}$$

which reduces to

$$u(0) \equiv u_0 = \epsilon^2 U \sum_1^N \mathscr{U}_n \equiv \epsilon^2 U \mathscr{U} \tag{5.1b}$$

on the axis. The parameter $\mathscr{U}$ is plotted in figure 3. The velocity profile u/u_0 is plotted as a function of the dimensionless radius Kr (which is independent of R) in figure 4 for $K = (N+\frac{3}{4})\pi$, $N = 1, 2, 4, 8$. The profiles for $N = 2$, 4, 8 are remarkably similar for $Kr < 2\pi$ and are quite flat for small Kr;

$$1 - |u/u_0| < 0{\cdot}01\,(0{\cdot}07)$$

for $Kr < 1{\cdot}6\,(2.4)$. The profile for $N = 16$ (not plotted) lies slightly below that for $N = 8$ in $Kr < 5{\cdot}6$ but has approximately the same zero at $Kr = 5.6$. These relatively blunt profiles contrast with the axial-velocity profile for a dipole in an unbounded rotating flow, for which (Miles 1969)

$$u/u_0 = J_0(Kr), \quad u_0 = O(1/|x|) \quad (x \to \infty), \tag{5.2a, b}$$

and u/u_0 decreases to $0{\cdot}5$ at $Kr = 1{\cdot}6$ and to 0 at $Kr = 2{\cdot}4$.

The energy-flux and impulse parameters $\mathscr{E}^{(w)}$, $\mathscr{E}^{(-)}$, and $\mathscr{P}^{(-)}$ are plotted in figures 5 and 6.

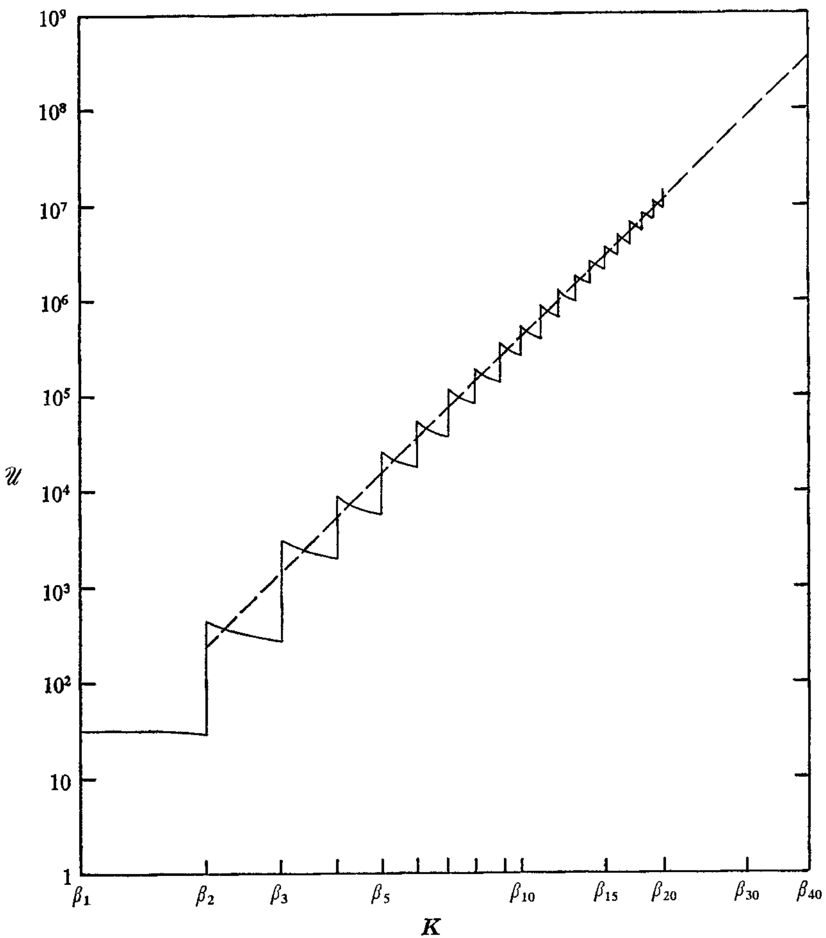

FIGURE 3. The peak ($r = 0$) upstream velocity $u_0 = \epsilon^2 \mathscr{U} U$ in the direction of motion of the dipole; see (5.1). The asymptotic extrapolation of (5.3a) is given by the dashed line.

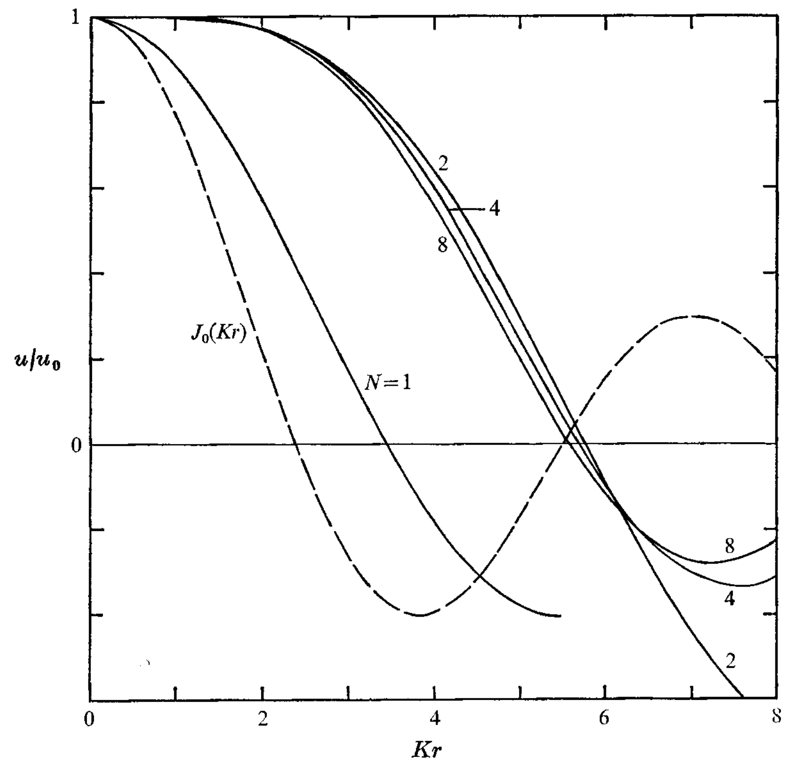

FIGURE 4. Upstream axial velocity distribution induced by dipole for $K/\pi = 1{\cdot}75,\ 2{\cdot}75,\ 4{\cdot}75,\ 8{\cdot}75$ ($N = 1, 2, 4, 8$), as given by (5.1). The dashed curve corresponds to an unbounded flow, for which $u_0/U = O(1/|x|)$ as $x \to -\infty$.

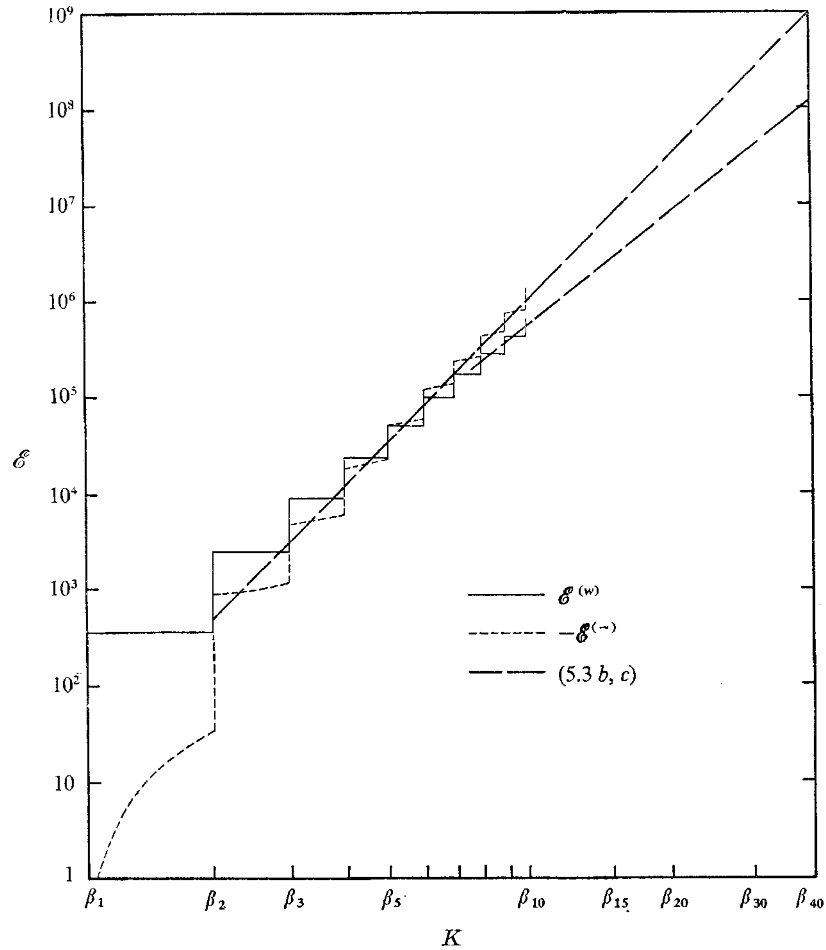

FIGURE 5. The upstream- and lee-wave-energy parameters, as defined by (4.7) and (4.9). The asymptotic extrapolations of (5.3*b*, *c*) are given by the dashed lines.

Asymptotic approximations

Asymptotic approximations for large K may be obtained by invoking the approximations (3.6) and (3.7) in (4.2), (4.7), (4.9), and (4.12) and then converting the summations to integrals.† Only the integral for $\mathscr{E}^{(w)}$ is tractable; however, the integrals for $\mathscr{U}$, $\mathscr{E}^{(-)}$, and $\mathscr{P}^{(-)}$ do yield heuristic extrapolations up to constants of proportionality. Evaluating these constants through numerical extrapolation of the calculated results for $N = 10(1)40$, we obtain

$$\left.\begin{aligned} \mathscr{U} &\sim 0{\cdot}011K^5, & \mathscr{E}^{(w)} &\sim \tfrac{1}{2}K^4, \\ \mathscr{E}^{(-)} &\sim -0{\cdot}032K^5, & \mathscr{P}^{(-)} &\sim 0{\cdot}10K^4 \end{aligned}\right\} \quad (K \to \infty). \qquad \begin{aligned}(5.3a,b)\\(5.3c,d)\end{aligned}$$

† This conversion, from summation over a continuous spectrum to integration over a continuous spectrum, is mathematically delicate in consequence of the singularities at $K = \beta_n$; however, the difficulties can be overcome and do not appear to merit a detailed discussion in the present context.

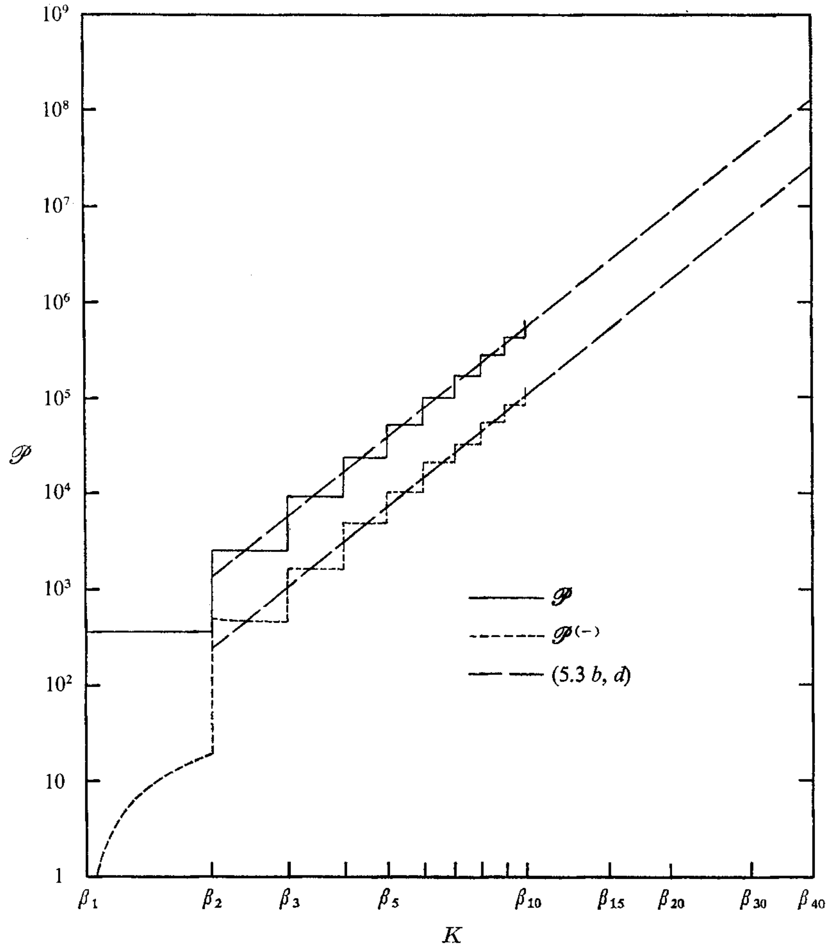

FIGURE 6. The upstream-impulse parameter, as defined by (4.12), and the total-impulse parameter $\mathscr{P} \equiv \mathscr{E}^{(w)}$. The asymptotic extrapolations of (5.3*b*, *d*) are given by the dashed lines.

[The convergence of the series for the K-scaled parameters – e.g. $\mathscr{U}/K^5$ – as $N \to \infty$ is painfully slow and resembles that for similar problems (in diffraction theory) in which a discrete spectrum tends to a continuous spectrum. The numerical coefficients in (5.3*a, c, d*) were determined by graphical extrapolation of plots of the K-scaled parameters versus $1/N$.] We remark that the significance of the divergence of $\mathscr{E}^{(-)}/\mathscr{E}^{(w)} \sim -0{\cdot}064K$ as $K \to \infty$ is limited by the divergence of the downstream perturbation energy as $t \to \infty$ [see discussion following (4.4)].

We use these last approximations to establish limiting results for unbounded rotating flow past an obstacle of *equivalent* radius a, defined such that (see remarks in § 1 above)

$$\epsilon \equiv (a/R)^3 \equiv \delta^3. \tag{5.4}$$

Introducing $\mathscr{k}$ from (1.1), we obtain the following results in the asymptotic limit (ii), $\delta \downarrow 0$ with $\mathscr{k}$ fixed (or, equivalently, $K \uparrow \infty$ with $\mathscr{k}$ fixed).

$$u_0/U \sim 0{\cdot}011\mathscr{k}^5\delta, \tag{5.5}$$

$$dE^{(-)}/dT \sim -0{\cdot}032k^5\delta\hat{E}_0 \quad (5.6a)$$

$$\doteqdot -3\hat{E}_0(u_0/U), \quad (5.6b)$$

$$C_D \equiv (\tfrac{1}{2}\pi\rho U^2a^2)^{-1}D \sim \tfrac{1}{2}\ell^4, \quad (5.7)$$

and

$$dP^{(-)}/dT \sim 0{\cdot}20D. \quad (5.8)$$

We emphasize that the significance of these approximations for finite obstacles is limited by the approximation $\mathscr{D}_n = 1$, which implies the restriction (4.15). Letting $\ell \to \infty$ in (4.14), we obtain $\mathscr{D}_n = O(\ell^{-2})$, which implies a corresponding reduction in the order of magnitude of each of (5.5)–(5.7) for $\ell \to \infty$.

Considering a sphere of $\frac{1}{2}$ in. diameter in a tube of 12 in. diameter, as in Maxworthy's (1970) experiments, we obtain $u_0/U = 0{\cdot}01$, $0{\cdot}02$, and $0{\cdot}06$ at $\ell = 1{\cdot}74$, $2{\cdot}16$, and $2{\cdot}61$ (the three values for which Maxworthy reports u_0 as a function of axial distance upstream of the sphere). These values are outside the range of Maxworthy's data but are much smaller than the observed velocities in $5 < |x|/a < 20$. On the other hand, the observed velocities do match the theoretical calculations based on Long's hypothesis and an unbounded flow after fitting an ellipsoid to the observed upstream-separation bubble (Miles 1969).† Moreover, the first zero in the observed profiles is at $Kr = 2{\cdot}5$–$2{\cdot}6$, which is close to the theoretical value of $Kr = 2{\cdot}4$ for unbounded inviscid flow and far from the value of $Kr = 5{\cdot}5$–$5{\cdot}7$ implied by the multi-mode profiles of figure 4 (see above).

This last comparison suggests that the columnar disturbance induced by a small obstacle in a large tube ($\delta \ll 1$) is negligible for moderate values of ℓ. Nevertheless, (5.5) does imply that u_0/U ultimately increases sharply with ℓ (e.g., $u_0/U = \frac{1}{3}$ and $dE^{(-)}dT \doteqdot -\hat{E}_0$ for $\delta = \frac{1}{24}$ and $k = 3{\cdot}21$). The perturbation calculation on which (5.5) and (5.6) are based and the approximation $\mathscr{D}_n \doteqdot 1$ then break down, but the results do suggest that the columnar disturbance induced by even a small obstacle could be responsible for blocking.

Finally, we infer from (5.5) and (5.8) that: on the one hand, upstream influence is absent in an unbounded flow in the sense (and this is the sense of Long's hypothesis and of § 6.2 in McIntyre's paper) that the axial velocity vanishes like δ; on the other hand, upstream influence exerts a finite effect in an unbounded flow in the sense (and this is essentially Benjamin's sense) that approximately one-fifth of the impulse flux associated with the wave drag appears in the upstream flow.

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† Basing the dipole strength on this ellipsoid, rather than on the actual sphere, increases the preceding values of u_0/U, but not by more than a factor of 2.

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